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LEVEL II



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6 THE EXTREMAL LARGE DEVIATION RATE
OF THE F-STATISTIC.

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## ABSTRACT

The extremal large deviation rate of the F-statistic over a large class is determined. It is shown that the Bahadur efficiency of the F-test relative to Moses' rank test for equality of variances is zero at every alternative when the distribution of the underlying observations is unspecified.



Key words: Bahadur efficiency, extremal large deviations, F-statistic, nonparametric tests.


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I. Introduction. Extremal large deviation rates measure the slowest rate of convergence to zero of certain sequences of probabilities. When these probabilities arise from a sequence of test statistics, extremal large deviation rates find application in the computation of asymptotic relative efficiency (ARE) of tests, such as Bahadur's definition of ARE.

The extremal large deviation rate of only a few common statistics is known. Here we will determine the extremal large deviation rate of the F-statistic.

Before precisely defining extremal large deviation rates, we will discuss Bahadur's definition of ARE, in which it finds application. Let $(X, A)$ be a measurable space. Let $\underset{\sim}{X} = \{X_i\}_{i=1}^{\infty}$ be a sequence of random variables on $(X, A)$, and let $P$ be a probability measure on $(X, A)$ such that $\underset{\sim}{X}$ is a sequence of independent identically distributed random variables with common distribution $P_F$. Let C and C' be disjoint non-empty collections of distributions, and consider the problem of testing

$$H_0: P \in C \text{ vs. } H_a: P \in C'.$$

Typically, decision rules are of the form "reject $H_0$ if $S_n$ exceeds some specified value a", where $S_n$ is a measurable real valued function of $\{X_i\}_{i=1}^{n}$; that is $\underset{\sim}{S} = \{S_i\}_{i=1}^{a}$ is a sequence of test statistics. Call this test $T_s(a)$.

Let $\underset{\sim}{S}' = \{S_i'\}_{i=1}^{\infty}$ be another sequence of test statistics. Call the test that rejects $H_0$ when $S_n'$ exceeds a specified value, say a', $T_{s'}(a')$. The test $T_{s'}(a')$ is a competitor to the test $T_s(a)$. We choose between these tests by considering their ARE.

The most important definitions of ARE are due to Pitman, (1949), Hodges-Lehmann (1956), and Bahadur (1960). Under mild regularity conditions, evaluation of the Bahadur efficiency of $T_S(a)$ relative to $T_{S'}(a')$ requires the determination of extremal large deviations rates of $\{S_i\}_{i=1}^{\infty}$ and $\{S'\}_{i=1}^{\infty}$.

We now make precise the concept of an extremal large deviation rate.

Definition 1: If

$$W(\underset{\sim}{S}, C, a) = -\lim_{n\to\infty} n^{-1} \log \sup_{P\epsilon C} P(S_n \geq a)$$

is well defined, we say that $W(\underset{\sim}{S}, C, a)$ is the extremal large deviation rate at a deviation of a for the sequence $\underset{\sim}{S}$ under the class of probability models C.

That is, $W(\underset{\sim}{S}, C, a)$ is a measure of the rate at which

$$\sup_{P\epsilon C} P(S_n \geq a)$$

converges to zero.

Let C be such that for every $P \epsilon C$,

$$P(S_n \to 0) = 1.$$

Consider the test $T_S(a)$, where $a > 0$. Now $F_n^*(a) = \sup_{P\epsilon C} P(S_n \geq a)$ is the significance level of $T_S(a)$, and $L_n = F_n^*(S_n)$ is the usual observed significance level. If $P \epsilon C$, $P(L_n \to 0) = 0$. If $P \notin C$ (in particular, if $P \epsilon C'$, the alternative class), then $P(L_n \to 0) = 1$. In typical cases, $L_n$ converges to zero exponentially fast at non-null P.

Definition 2: The half slope of $T_S(\ )$ at non-null P is given by

$$\kappa(P) = -\lim_{n\to\infty} n^{-1} \log L_n [P]$$

when the above is well defined.

We may interpret $\rho(P)$ as a measure of the rate at which the observed significance level tends to zero, with large values indicating more rapid convergence. Bahadur has exploited this property to define a measure of ARE, described below.

Definition 3: Let $T_S(\cdot)$ and $T_{S'}(\cdot)$ be as defined earlier. At non-null P, let $T_S(\cdot)$ have half slope $\rho(P)$ and let $T_{S'}(\cdot)$ have half slope $\rho'(P)$. The Bahadur efficiency of $T_S(\cdot)$, relative to $T_{S'}(\cdot)$, at non-null P is given by $\rho(P)/\rho'(P)$.

The following theorem, due to Bahadur, relates the concepts of half slope and extremal large deviation rates.

Theorem 1: (Bahadur, 1960). Let $P(S_n \to a(P)) = 1$ at non-null P. Let $W(\underset{\sim}{S}, C, t)$ be well defined for t in a neighborhood of a(P) and continuous at $t = a(P)$. Then $\rho(P) = W(\underset{\sim}{S}, C, a(P))$.

Consider the case $S_n = n^{-1} \sum_{i=1}^{n} X_i$, where C consists of a single distribution F and $\int x dF(x) = 0$. Using Chernoff's Theorem, stated below, evaluation of the half slope of $\underset{\sim}{S}$ at a fixed non-null distribution is straightforward.

Theorem 2: (Chernoff, 1952). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables. Let $\phi(t) = Ee^{tX_1}$, not necessarily finite for $t \neq 0$. Then for every real a,

$$\lim_{n\to\infty} n^{-1} \log P(n^{-1} \sum_{i=1}^{n} X_i \geq a) = \inf_{t\geq 0} \log \phi(t)e^{-at}.$$

Hence, at non-null G, the half slope of $\underset{\sim}{S}$ equals $-\inf_{t\geq 0} \log \psi(t)e^{-\alpha t}$ where $\psi(t) = \int e^{tx} dF(x)$ and $\alpha = \int x dG(x)$.

## II. Large Deviation Rates of the F-Statistic

Extremal large deviation rates over a large class have only been evaluated in a few instances. For example, if the statistic of interest is a linear rank statistic, a straightforward method exists. See Klotz (1965), who evaluates the half slope of the Wilcoxon rank sum statistic, as well as other linear rank statistics.

One of the few non-distribution-free statistics for which the extremal large deviation rate has been studied is the t-statistic.

Theorem 3: (Jones and Sethuraman, 1978). Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be two sequences of mutually independent identically distributed random variables, with common distribution F and joint distribution $P_F$. Let

$$\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i,\ \bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i,$$

$$s_{x,n}^2 = n^{-1}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2, \text{ and}$$

$$s_{y,n}^2 = n^{-1}\sum_{i=1}^{n} Y_i^2 - (\bar{Y}_n)^2.$$

Define the one-and two-sample t-statistics, $T_{1,n}$ and $T_{2,n}$ respectively, as

$$T_{1,n} = \bar{X}_n/(s_{x,n}^2)^{\frac{1}{2}} \text{ and}$$
$$T_{2,n} = (\bar{X}_n - \bar{Y}_n)/((s_{x,n}^2 + s_{y,n}^2)/2)^{\frac{1}{2}}$$

where expressions of the form r/0 are interpreted as

$$\begin{Bmatrix} +\infty & \text{if} & r > 0 \\ 0 & \text{if} & r = 0 \\ -\infty & \text{if} & r < 0 \end{Bmatrix}.$$

Let $C_1 = \{F: F \text{ is symmetric about } 0\}$ and let $C_2 = \{(F, G): F = G\}$. Then for every $a > 0$, $j = 1$ or $2$, $W(\{T_{j,n}\}_{n=1}^{\infty}, C_j, ja) = jI(\frac{1}{2}(1 + a/(1 + a^2)^{\frac{1}{2}}), \frac{1}{2})$, where for $0 < p_1, p_2 < 1$, $I(p_1, p_2) = p_1 \log \frac{p_1}{p_2} + (1 - p_1) \log \frac{1 - p_1}{1 - p_2}$. Further, this rate of convergence is achieved by a t-statistic based on symmetric Bernoulli observations. That is, the t-statistic converges most slowly when based on symmetric Bernoulli observations.

<u>Definition 4</u>: Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$, $s_{x,n}^2$, $s_{y,n}^2$, and indeterminant forms be as in Theorem 3. Let the <u>F-statistic</u>, $F(n)$, be given by $s_{x,n}^2/s_{y,n}^2$. Let the <u>shifted F-statistic</u>, $F_s(n)$, equal $F(n) - 1$.

It is reasonable to base a test of equality of variances, assumed to exist, on $F_s(n)$, with large values indicating rejection. Under the null hypothesis, $F_s(n) \to 0$ a.s. At an alternative $H_a$, $F_s(n) \to a(H_a)$ a.s., where $a(H_a)$ is assumed positive. We will now determine the large deviation rate of $F_s(n)$ under certain parametric assumptions.

<u>Theorem 4</u>: Let $F$ assign mass $\frac{1}{2}$ each to $\pm 1$. That is, $F$ is a symmetric Bernoulli distribution. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be mutually independent identically distributed random variables, with common distribution $F$. Let $F_s(n)$ be based on $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$. Then for any $a > 0$,

$$-\lim_{n\to\infty} n^{-1} \log P(F_s(n) \geq a) = I(\tfrac{1}{2}(1 + (a/1 + a)^{\frac{1}{2}}), \tfrac{1}{2})$$

where $I(\cdot, \cdot)$ is as in Theorem 3.

<u>Proof</u>: For any real $a$,

$$P(F_s(n) \geq a) = P(s_{x,n}^2/s_{y,n}^2 \geq 1 + a) = P((1 - \bar{X}_n^2)/(1 - \bar{Y}_n^2) \geq 1 + a) =$$

$$P((1 + a)\bar{Y}_n^2 - \bar{X}_n^2 \geq a) \leq P(\bar{Y}_n^2 \geq a/(1 + a)),$$

By Chernoff's Theorem,

1 $$\left(\lim_{n\to\infty} n^{-1} \log P(Y_n^2 \geq a/(1 + a)\right) = \left(-I(\tfrac{1}{2}(1 + (a/a + 1)^{\frac{1}{2}}), \tfrac{1}{2}\right).$$

Choose and fix $\varepsilon$, $1 > \varepsilon > 0$. Now $P(F_s(n) \geq a) = P((1 - \bar{X}_n^2)/(1 - \bar{Y}_n^2) \geq 1 + a) \geq$ $P(\bar{Y}_n^2 \geq (a + \varepsilon)/(a + 1),\ \bar{X}_n^2 < \varepsilon) = P(\bar{Y}_n^2 \geq (a + \varepsilon)/(a + 1))\ P(\bar{X}_n^2 < \varepsilon)$.

But $\lim_{n\to\infty} \log P(\bar{X}_n^2 < \varepsilon) = 0$;

Hence
$$\lim_{n\to\infty} \lim_{n\to\infty} n^{-1} \log P(F_s(n) \geq a) \geq$$
$$\lim_{n\to 0} \lim_{n\to\infty} n^{-1} \log P(\bar{Y}_n^2 \geq (a + \varepsilon)/(a + 1)) =$$
$$\lim_{\varepsilon\to 0} -I(\tfrac{1}{2}(1 + ((a + \varepsilon)/(a + 1))^{\frac{1}{2}}), \tfrac{1}{2}) =$$
$$I(\tfrac{1}{2}(1 + ((a + \varepsilon)/(a + 1))^{\frac{1}{2}}), \tfrac{1}{2}).$$

By lines 1 and 2,

$$-\lim_{n\to\infty} n^{-1} \log P(F_s(n) \geq a) = I(\tfrac{1}{2}(1 + (a/a + 1)^{\frac{1}{2}}), \tfrac{1}{2}). \quad \square$$

Theorem 5: Let $F$ be a normal distribution, with mean $\mu$ and variance $\sigma^2 > 0$. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be mutually independent identically distributed random variables with common distribution $F$. Let $F_s(n)$ be based on $\{X_i\}_{i=1}^{n}$, $\{Y_i\}_{i=1}^{\infty}$. Then for any $a \geq 0$,

$$-\lim_{n\to\infty} n^{-1} \log P(F_s(n) \geq a) = \log(2(a + 1)^{\frac{1}{2}}/(a + 2)).$$

Proof: Without loss of generality, let $\mu = 0$, $\sigma^2 = 1$. Let $\{U_i\}_{i=1}^{\infty}$, $\{V_i\}_{i=1}^{\infty}$ be two sequences of mutually independent identically distributed random variable with common $\chi_1^2$ distribution. Then for every $a \geq 0$,

$$P(F_s(n) \geq a) =$$
$$P\left(\sum_{i=1}^{n-1} U_i / \sum_{i=1}^{n-1} V_i \geq a + 1\right) =$$
$$P\left(\sum_{i=1}^{n-1} U_i \geq \sum_{i=1}^{n-1} (a + 1)V_i\right) =$$
$$P\left(\sum_{i=1}^{n-1} (U_i - (a + 1)V_i)/(n - 1) \geq 0\right).$$

By Chernoff's Theorem

$$\lim_{n\to\infty} n^{-1} \log P\left(\sum_{i=1}^{n-1} (U_i - (a+1)V_i)/(n-1) \ge 0\right) =$$

$$\inf_{t\ge 0} \log(1 + 2at - 4(a+1)t^2)^{-\frac{1}{2}} =$$

$$\log(2(a+1)^{\frac{1}{2}}/(a+2)). \square$$

In Table 1, we display $I(\frac{1}{2}(1 + (a/(a+1))^{\frac{1}{2}}), \frac{1}{2})$ and $-\log(2(a+1)^{\frac{1}{2}}/(a+2))$ for selected values of $a > 0$. We see that $F_s(n)$ converges to zero more rapidly under some alternatives when the null distribution of the underlying observations is normal than if the null distribution of the underlying observations is symmetric Bernoulli. Unlike the t-statistic, the F-statistic does not converge most slowly when the underlying observations have a symmetric Bernoulli distribution.

III. Extremal Large Deviation Rates of the F-statistic. Before dealing with the F-statistic, which is used to test for equality of variances, we will consider a related statistic used for testing the equality of means of positive random variables.

Definition 4: Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be mutually independent identically distributed random variables with common distribution $F \in G$, where $G = \{F: \int_0^{\infty} dF = 1,\ 0 < \int_0^{\infty} x\,dF(x) < \infty\}$. Define the ratio statistic, $R_n$, as $(\bar{X}_n/\bar{Y}_n) - 1$, where $\bar{X}_n$, $\bar{Y}_n$, and indeterminate forms are as in Theorem 3.

The statistic $R_n$ can be used to test $H_0$: $EX_1 = EY_1$. Under $H_0$, $R_n \to 0$ a.s. Under a non-null F, $R_n \to a(F)$ a.s. Without loss of generality, assume $a(F) > 0$.

Theorem 6. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$, $G$, and $R_n$ be as in Definition 4. Then for every $a > 0$,

$$W(R_n, G, a) = 0.$$

Proof: Let F be a fixed but unspecified distribution in G. Clearly,

$$0 \le W(R_n, G, a) \le -\lim_{n\to\infty} n^{-1} \log P_F(R_n \ge a).$$

Let $\varepsilon > 0$ be given and fixed. Let $\alpha = \varepsilon/\log(1 + (a^2)/4(a + 1))$, and let F be a Gamma $(\alpha, 1)$ distribution. Then

$$\lim_{n\to\infty} n^{-1} \log P_F(R_n \ge a) = \lim_{n\to\infty} n^{-1} \log P_F(\sum_{i=1}^{n} X_i \ge (a + 1) \sum_{i=1}^{n} Y_i) =$$

$$\lim_{n\to\infty} n^{-1} \log P_F(\sum_{i=1}^{n} X_i - (a + 1)Y_i \ge 0) = \log \inf_{t\ge 0} \psi(t)\,\psi(-(a + 1)t),$$

by Chernoff's Theorem, where $\psi(t) = (1 - t)^{-\alpha}$.

By elementary methods, $\log \inf_{t\geq 0} \psi(t)\,\psi(-(a+1)t) = -\varepsilon$. The result follows, taking the limit as $\varepsilon \to 0$. □

Before determining the extremal large deviation rate of the F-statistic, some preliminary results are needed.

Lemma 1. Let x tend to zero from above. Then $I(\frac{1}{2}(1+x), \frac{1}{2}) = x^2/2 + 0(x^2)$.

Proof: Expanding $I(\frac{1}{2}(1+x), \frac{1}{2})$ as a Taylor series, the result follows.

Lemma 2 (Jones and Sethuraman, 1978). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. symmetric random variables.

Then for every sample size n and $b \in (0, 1)$

$$P\Big(\sum_{i=1}^{n} X_i / \Big(\sum_{i=1}^{n} X_i^2\Big)^{\frac{1}{2}} \geq n^{\frac{1}{2}} b\Big) \leq \exp(-nI(\tfrac{1}{2}(1+b), \tfrac{1}{2})).$$

We are now ready to determine the extremal large deviation rate of the F-statistic over appropriate classes.

Theorem 7. Let $\mathcal{F} = \{F\colon\ F \text{ is symmetric about } 0, \int x^2 dF < \infty\}$. Let $\{U_i\}_{i=1}^{\infty}, \{V_i\}_{i=1}^{\infty}$ be two sequences of mutually independent identically distributed random variables with common distribution F, where $F \in \mathcal{F}$. Let $F_s(n)$ be based on $\{U_i\}_{i=1}^{\infty}$; $\{V_i\}_{i=1}^{\infty}$. Then for every $a \geq 0$,

$$W(\underset{\sim}{F}_s(n), \mathcal{F}, a) = 0.$$

Proof: Choose and fix $\varepsilon > 0$. It is sufficient to find an F in $\mathcal{F}$ such that $\lim_{n\to\infty} n^{-1} \log P_F(F_s(n) \geq a) \geq -\varepsilon$.

Let $\alpha$ be chosen as in Theorem 6. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be mutually independent identically distributed random variables such that $X_1^2$ has a

Gamma $(\alpha, 1)$ distribution. Let $\{\delta_i\}_{i=1}^{\infty}$, $\{\eta_i\}_{i=1}^{\infty}$ be mutually independent identically distributed symmetric Bernoulli random variables, independent of $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$. Let $F_s(n)$ be based on $\{U_i\}_{i=1}^{\infty} = \{\eta_i X_i\}_{i=1}^{\infty}$ and

$$\{V_i\}_{i=1}^{\infty} = \{\delta_i Y_i\}_{i=1}^{\infty}.$$

Let $Z = 1 - (a + 1)(\sum_{i=1}^{n} Y_i^2 / \sum_{i=1}^{n} X_i^2)$. For any $a > 0$,

$$P_F(F_s(n) \ge a) = P_F((a + 1)(\sum_{i=1}^{n} \delta_i Y_i)^2 - (\sum_{i=1}^{n} \eta_i X_i)^2 \ge n(\sum_{i=1}^{n} (a + 1)Y_i^2 - X_i^2)) =$$

$$P_F((\sum_{i=1}^{n} \eta_i X_i)^2 \le n[\sum_{i=1}^{n} (a + 1)Y_i^2 - \sum_{i=1}^{n} X_i^2] + (a + 1)(\sum_{i=1}^{n} \delta_i Y_i)^2) \ge$$

$$P_F((\sum_{i=1}^{n} \eta_i X_i)^2 \le n \sum_{i=1}^{n} X_i^2\, Z) = E_F P_F((\sum_{i=1}^{n} \eta_i X_i)^2 \le n \sum_{i=1}^{n} X_i^2\, Z \,|\, \{X_i, Y_i\}_{i=1}^{\infty}) =$$

$$1 - 2\, EP_F((\sum_{i=1}^{n} \eta_i X_i)/(n \sum_{i=1}^{n} X_i^2)^{\frac{1}{2}} \ge Z^{\frac{1}{2}} \,|\, \{X_i, Y_i\}_{i=1}^{\infty}) \ge \tag{3}$$

$$E(1 - 2 \exp(-nI(\tfrac{1}{2}(1 + Z^{\frac{1}{2}}), \tfrac{1}{2})\{X_i, Y_i\}_{i=1}^{\infty}) \ge \tag{4}$$

$$\rho(-n\, I(\tfrac{1}{2}(1 + Z^{\frac{1}{2}}), \tfrac{1}{2})\, I(Z \ge 2/n) \,|\, \{X_i, Y_i\}_{i=1}^{\infty}) \ge$$

$$(1 - 2 \exp(-nI(\tfrac{1}{2}(1 + (2/n)^{\frac{1}{2}}), \tfrac{1}{2})))P_F(Z \ge 2/n),$$

where (4) follows from (3) by Lemma 2.

By Lemma 1, $1 - 2 \exp(-n\, I(\tfrac{1}{2}(1 + (2/n)^{\frac{1}{2}}), \tfrac{1}{2})$ is asymptotic to $(1 - 2 \exp(-1))$. By direct substitution,

$$P_F(Z \ge 2/n) = P_F(\sum_{i=1}^{n} X_i^2 / \sum_{i=1}^{n} Y_i^2 \ge (a + 1)n/(n - 2)).$$

Hence, $\lim_{n\to\infty} n^{-1} \log P_F(F_s(n) \ge a) \ge \lim_{n\to\infty} n^{-1} \log P_F(\sum_{i=1}^{n} X_i^2 / \sum_{i=1}^{n} Y_i^2 \ge a + 1) = -\varepsilon$.

result follows. □

One is tempted to conjecture the existence of a subclass of $F$, say $F^*$, such that $W(\underset{\sim}{F}_s(n), F^*, a) > 0$. No such class has been discovered, although a number of candidates have been ruled out. Notice that the construction in Theorem 7 is scale invariant. By suitable scaling, this construction can be made to satisfy any condition on the absolute first moment, the second moment, or the pth percentile of the underlying distribution. Restricting the underlying observations to the class of symmetric and bounded distributions does not yield a non-zero extremal large deviation rate, as can be seen below.

Theorem 8: Let $H = \{F\colon F$ is symmetric about zero, and has support in $[-1, +1]\}$. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be two sequences of mutually independent identically distributed random variables with common distribution F, where $F \in H$. Let $F_s(n)$ be based on $\{X_i, Y_i\}_{i=1}^{\infty}$. Then for every $a \geq 0$, $W(\underset{\sim}{F}_s(n), H, a) = 0$.

Proof: Proceeding as before, let $\varepsilon > 0$ be chosen and fixed.

Let H be the distribution that assigns mass $e^{-\varepsilon}$ to 0 and is uniformly distributed over $(-1, +1)$ with probability $(1 - e^{-\varepsilon})$. Then for every $a \geq 0$,

$$\lim_{n\to\infty} n^{-1} \log P_H(F_s(n) \geq a) \geq$$
$$\lim_{n\to\infty} n^{-1} \log P_H(F_s(n) = +\infty) =$$
$$\lim_{n\to\infty} n^{-1} \log P_H(s^2_{x,n} > 0,\ s^2_{y,n} = 0) =$$
$$\lim_{n\to\infty} n^{-1}\{\log P(s^2_{x,n} \neq 0) + \log P(s^2_{y,n} = 0)\} =$$
$$\lim_{n\to\infty} n^{-1}\{\log(1 - e^{-n\varepsilon}) + \log(e^{-n\varepsilon})\} =$$
$$-\varepsilon. \quad \square$$

## IV. Efficiencies of Tests for Equality of Variances

Consider the problem of testing for equality of variances of two populations, with underlying distributions unspecified. The usual procedure is to reject the null hypothesis, equality of variances, if the statistic $F_s(n)$ exceeds some specified value. Call this test $T_1$. Let us consider a competitor to $T_1$.

Moses (1963) has proposed a distribution-free test for equality of variances. Moses recommends randomly allocating $\{X_i\}_{i=1}^{kn}$ into n subsamples of size k, and calculating sample variances. This is repeated with $\{Y_i\}_{i=1}^{kn}$, and the two collections of sample variances are then compared by means of the Wilcoxon rank sum test. Call this test $T_2$.

It is known that the half slope of the Wilcoxon rank sum test is positive. Hence, the half slope of $T_2$ is also positive. By Theorem 7, the half slope of $T_1$ is zero at every alternative. Hence, at every alternative, the Bahadur efficiency of the F-test, relative to Moses' test, is zero.

While the t-test performs relatively well against distribution free competitors, the F-test performs as poorly as possible. One is tempted to propose a "t-test for variances" of the form

$$S_n = (s^2_{x,n} - s^2_{y,n})/(s^4_{x,n} + s^4_{y,n})^{\frac{1}{2}}.$$

The statistic $S_n$ can be shown to be a monotone increasing function of $F_s(n)$ if $F_s(n) > 0$ and a monotone decreasing function of $F_s(n)$ if $F_s(n) < 0$. Hence, the half slope of $S_n$ is zero at every alternative, just as the half slope of $F_s(n)$ is.

Table 1: Large Deviation Rates of the F-statistic

| Deviation a | Normal Observations $-\log(2(a+1)^{\frac{1}{2}}/(a+2))$ | Bernoulli Observations $I(\frac{1}{2}(1+(a/a+1)^{\frac{1}{2}}), \frac{1}{2}$ |
|---|---|---|
| .05 | .00030 | .02400 |
| .10 | .00114 | .04617 |
| .15 | .00244 | .06672 |
| .20 | .00415 | .08582 |
| .25 | .00621 | .10363 |
| .30 | .00858 | .12029 |
| .35 | .01122 | .13591 |
| .40 | .01652 | .15058 |
| .45 | .01716 | .16440 |
| .50 | .02041 | .17744 |
| .55 | .02382 | .18977 |
| .60 | .02736 | .20144 |
| .65 | .03102 | .21251 |
| .70 | .03479 | .22304 |
| .75 | .03865 | .22305 |
| .80 | .04257 | .24259 |
| .85 | .04658 | .25169 |
| .90 | .05064 | .26038 |
| .95 | .05474 | .26869 |
| 1.00 | .05889 | .27665 |
| 1.25 | .08004 | .31188 |
| 2.50 | .10147 | .34102 |
| 2.75 | .12281 | .36557 |
| 3.00 | .14384 | .38658 |

Table 1
(continued)

| Deviation | Normal Observations | Bernoulli Observations |
|---|---|---|
| a | $-\log(2(a+1)^{\frac{1}{2}}/(a+2))$ | $I(\frac{1}{2}(1+(a/a+1)^{\frac{1}{2}}),\frac{1}{2}$ |
| 3.50 | .18455 | .42072 |
| 4.00 | .22314 | .44737 |
| 5.00 | .29389 | .48651 |
| 6.00 | .35688 | .51406 |
| 7.00 | .41334 | .53456 |
| 8.00 | .46436 | .55052 |
| 9.00 | .51083 | .56332 |
| 10.00 | .56358 | .57618 |
| 11.00 | .59281 | .58264 |
| 12.00 | .62934 | .59014 |
| 13.00 | .66344 | .59660 |
| 14.00 | .69537 | .60225 |
| 15.00 | .72542 | .60721 |
| 20.00 | .85351 | .62526 |
| $\infty$ | $\infty$ | log 2 |

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20. ABSTRACT

The extremal large deviation rate of the F-statistic over a large class is determined. It is shown that the Bahadur efficiency of the F-test relative to Moses' rank test for equality of variances is zero at every alternative when the distribution of the underlying observations in unspecified.